Stratification frequency: a new insight

Guillaume Riflet
MARETEC IST

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1 Purpose

While trying to reproduce the PAPA station test-case GOTM results with MOHID, we tumbled into a different conception of the Brunt-Vaisalla frequency when using the pressure correction to the UNESCO density Equation Of State (EOS) [1]. Our original version was

$$N^2 = -\frac{g}{\rho_0}\frac{\partial \rho}{\partial z} \tag{1}$$

as is stated in the Cushman-Roisin [1] It worked fine with without pressure correction on the EOS. However, turning the pressure on caused an extra stratification leading to differences over 3 degrees in the SST of the PAPA station test-case. Apparently, GOTM didn't really use the pressure correction and would only use a different reference pressure. But this isn't exactly so, and this work pin-points exactly what why the guys at GOTM use a correct expression for the stratification frequency, and why MOHID now has the same implementation. Furthermore, this work suggests that different stratification frequencies exist, one for each fluid present within the medium. This has direct implications in the evaluation of the stratification frequencies for tracers other than temperature. This version of the stabilization frequency doesn't requires the hydrostatic approximation as does Mellor's version [7]. Thus it seems interesting to print out a simple, and clear to understand, document relating this issue, especially because, since that pressure correction has been included, nobody seems to really look at it.

2 Review of the state-of-the-art

G. Mellor [7] introduces a new density gradient "suitable" for static stability i.e a "suitable" Brunt-Vaisalla frequency:

$$\frac{\partial \tilde{\rho}}{\partial z} \equiv \frac{\partial \rho}{\partial S}\frac{\partial S}{\partial z} + \frac{\partial \rho}{\partial \Theta}\frac{\partial \Theta}{\partial z} \tag{2}$$

$$N^2 \equiv -\frac{g}{\rho}\frac{\partial \tilde{\rho}}{\partial z} \tag{3}$$

where the term $\frac{\partial \rho}{\partial p}\frac{\partial p}{\partial z}$ in eq. (2) is excluded because

> *Physically one excludes the change in density a particule undergoes by an adiabatic change in depth and pressure; it is only non-adiabatic differences that are important to stability. G.M. [7]*

If we use the hydrostatic approximation $\frac{\partial p}{\partial z} = -\rho g$ and if we set $\frac{\partial \rho}{\partial p} = c^{-2}$ where c is speed of sound in the medium, then we obtain $\frac{\partial \rho}{\partial p}\frac{\partial p}{\partial z} = -\frac{\rho g}{c^2}$.

$$\frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial S}\frac{\partial S}{\partial z} + \frac{\partial \rho}{\partial \Theta}\frac{\partial \Theta}{\partial z} + \frac{\partial \rho}{\partial p}\frac{\partial p}{\partial z} = \frac{\partial \tilde{\rho}}{\partial z} - \frac{\rho g}{c^2} \tag{4}$$

Thus the Brunt-Vaisalla frequency becomes, according to G. Mellor

$$N^2 \equiv -\frac{g}{\rho}\frac{\partial \tilde{\rho}}{\partial z} = -\frac{g}{\rho}\left(\frac{\partial \rho}{\partial z} + \frac{\rho g}{c^2}\right) = -\frac{g}{\rho}\frac{\partial \rho}{\partial z} + \frac{g^2}{c^2} \tag{5}$$

He also states that

> *Another good approximation is $\frac{\partial \tilde{\rho}}{\partial z} = \frac{\partial \rho_\Theta}{\partial z}$. Thus the potential density function can be used to determine horizontal density gradients which drive horizontal motions and vertical density gradients which govern vertical mixing. [7]*

where $\rho_\Theta \equiv \rho\,(S,\,\Theta)$. $\boxed{\text{POM}}$ uses eq. (5) to calculate its N^2.
R. Hallberg 2005 [3] defines

$$N^2 = -\frac{g^2}{\alpha^2}\left(\frac{d\alpha}{dp} - \left.\frac{\partial \alpha}{\partial p}\right|_{\Theta,\,S}\right) \tag{6}$$

where α is the specific volume or thermal expansion coefficient.
Kantha and Clayson [4] define:

$$N^2 = g\alpha\left(\Gamma + \frac{dT}{dz}\right) - g\beta\frac{dS}{dz} \tag{7}$$

where $\alpha \equiv \frac{1}{\rho}\left.\frac{\partial\rho}{\partial T}\right|_{p,S}$, $\beta \equiv \frac{1}{\rho}\left.\frac{\partial\rho}{\partial S}\right|_{p,T}$ are the thermal expansion and haline contraction coefficients and where

$$\Gamma \equiv -\left.\frac{dT}{dz}\right|_{\sigma} = g\alpha T/c_p \tag{8}$$

is the adiabatic lapse rate, α' being the specific volume.
The latter expression is equivalent to the following one, in terms of potential temperature

$$N^2 = g\left(\alpha\frac{d\Theta}{dz} - \beta\frac{dS}{dz}\right) \tag{9}$$

as was suggested by Eden and Willebrand [2] or by McDougall and Jackett [5]. MOM4 uses the Accurate and computationally efficient algorithms of McDougall and Jackett [5]. ROMS uses the Jackett and McDougall 1995 EOS algorithm. It seems relevant to use potential temperature to calculate buoyancy effects in the stratification since it includes the adiabatic lapse rate effect.

3 The principle of Archimedes

We will show that equations (5), (6), (7) and (9) under the hydrostatic approximation, are equivalent, within a certain approximation, to a more generic and physically simpler to understand definition of static stability. It relies solely on the principle of Archimedes when small disturbances are applied to a Test Material Volume (TMV) at rest in a stable stratified fluid. The TMV could be any material where the thermodynamic's laws are valid. Let us undertake the following mental experience: consider a cork of density higher than surface seawater at rest in a stable stratified water column. Now consider a small vertical disturbance of it's mean rest state. Clearly, the restoring buoyant force will be equal to the balance between the cork and the displaced water weights (or densities) i.e. $Fb = \Delta\rho g$. Since the cork's density is the same of the surrounding medium when at rest, we can state that buoyancy is proportional to the density's vertical gradient and that that a normalized measure of the effect of buoyancy is

$$\frac{1}{\rho_0}\frac{\partial\rho}{\partial z} \tag{10}$$

where ρ is the medium's density and ρ_0 a reference density (e.g. the cork's density). This leads to equation (1).

However, when considering displacements of water parcels, these parcels no longer behave like a cork. They undergo expansion and contractions of their volume or they could exchange mass, heat etc... Thus, when applying a disturbance to a generic TMV, one has to enter into consideration its own density variations, $D\tilde{\rho}$. This wasn't required for the cork, because the cork hadn't any density variations. This is not the case for a generic TMV embedded in a given medium. The principle of Archimedes remains unchanged, and the generic measure of the buoyancy effect in a stably stratified fluid, called the square of the Brunt-Vaisalla frequency, is written as

$$N^2 = -\frac{g}{\rho_0}\left(\frac{d\rho}{dz} - \frac{d\tilde{\rho}}{dz}\right) \tag{11}$$

where the ˜ stands for the TMV state variables, thus $\tilde{\rho}$ is the TMV's density. Equation (11) is completely generic, has no approximations and yields the following discretization:

$$N_i^2 = -\frac{g}{\rho_0}\left(\frac{\rho\left(\Theta_{i+1}, S_{i+1}, p_{i+1}\right) - \rho\left(\Theta_i, S_i, p_i\right)}{z_{i+1} - z_i}\right. \tag{12}$$

$$\left. - \frac{\tilde{\rho}\left(\tilde{\Theta}_i, \tilde{S}_i, \tilde{p}_{i+1}\right) - \tilde{\rho}\left(\tilde{\Theta}_i, \tilde{S}_i, \tilde{p}_i\right)}{z_{i+1} - z_i}\right) \tag{13}$$

where i, indexes the state of rest and $i+1$, the perturbated state. If the TMV is incompressible and has no material variation of its density (like a cork), then equation (11) reduces to the original equation (1). We still need to relate, somehow, the TMV's density with the exterior density in order to make some use of eqs.(11) and (12). Figure (1) illustrates the perturbation applied to a seawater TMV initially at rest. It undergoes an adiabatic transformation and conserves its mass, hence only the pressure changes. How does the pressure changes? Well, it changes between its original value and the exterior value, it can even fluctuate in time. Thus we need another assumption:

$$\frac{\Delta h}{\Delta t} \ll c_s. \tag{14}$$

where c_s is the sound of speed. This allows the TMV to adiabatically adjust its inner pressure to the surrounding pressure as the speed of sound is far greater than the speed of the transformation. Hence, under this assumption, $\tilde{p} = p$ all along the transformation (see figure X). Furthermore, if it's a seawater TMV, then $\tilde{\Theta}_i = \Theta_i$, $\tilde{S}_i = S_i$ and the discretized eq.(11) simplifies to

$$N_i^2 = -\frac{g}{\rho_0}\left(\frac{\rho\left(\Theta_{i+1}, S_{i+1}, p_{i+1}\right)}{z_{i+1} - z_i} - \frac{\rho\left(\Theta_i, S_i, p_{i+1}\right)}{z_{i+1} - z_i}\right). \tag{15}$$

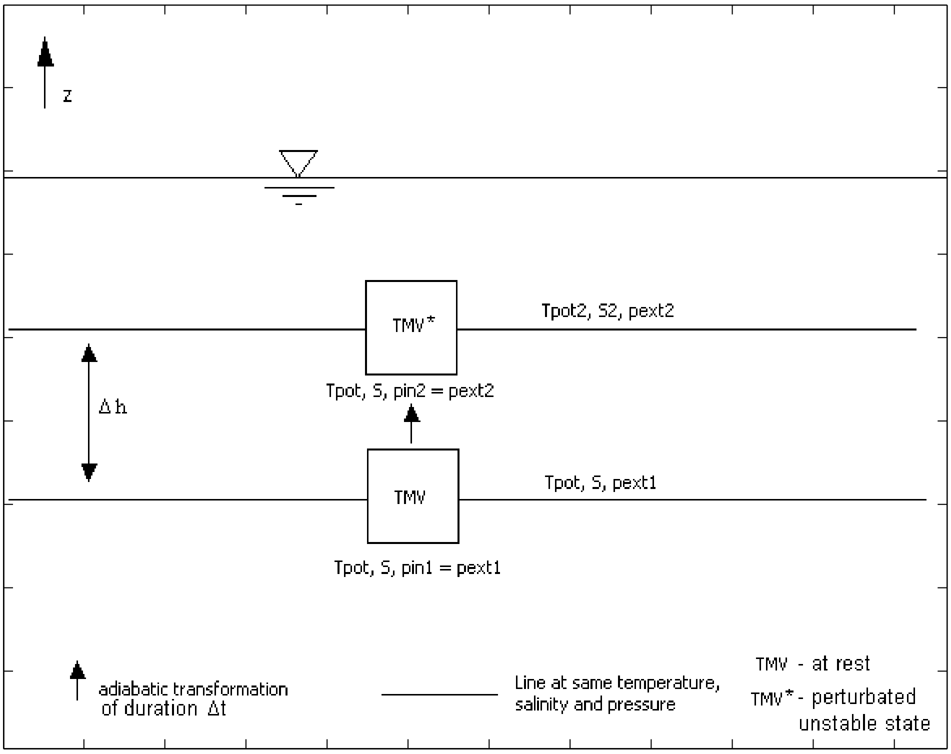

Figure 1: Illustration of an adiabatic and isohaline transformation of a seawater TMV, from a state of rest, to a perturbated state, away from equilibrium. The transformation is slow enough, so that compression forces have time to restore the pressure inside the TMV along the way.

This discretization is equivalent to the one used in $\boxed{\text{GOTM}}$, and, now, in $\boxed{\text{MOHID}}$. Also, in the absence of pressure correction, (15) reduces to the discretized form of (1).

4 The equivalence of the interpretation

Let us show the equivalence between this interpretation of the Brunt-Vaisalla frequency and Mellor's:
The variation of $\tilde{\rho} = \tilde{\rho}\left(\tilde{\Theta},\, \tilde{S},\, \tilde{p}\right)$ along a path s parameterized by t is stated in the material derivative

$$\frac{D\tilde{\rho}}{Dt} = \frac{\partial\tilde{\rho}}{\partial\tilde{\Theta}}\frac{D\tilde{\Theta}}{Dt} + \frac{\partial\tilde{\rho}}{\partial\tilde{S}}\frac{D\tilde{S}}{Dt} + \frac{\partial\tilde{\rho}}{\partial\tilde{p}}\frac{D\tilde{p}}{Dt} \tag{16}$$

$$= \frac{d\tilde{\rho}}{ds}\frac{ds}{dt} \tag{17}$$

The line path is vertical, the thermodynamical process is adiabatic ($d\tilde{\sigma} = 0$) and mass conservative ($D\tilde{S} = 0$). Consequently, $D\tilde{\Theta} = 0$. Thus equation (16) simplifies and writes

$$\frac{d\tilde{\rho}}{dz} = \frac{\partial\tilde{\rho}}{\partial\tilde{p}}\frac{D\tilde{p}}{Dt}\left(\frac{dz}{dt}\right)^{-1} \tag{18}$$

The variation of the residual scalar field $\rho = \rho\,(\Theta,\, S,\, p)$ along a path s per line element ds, is stated as

$$\frac{d\rho}{ds} = \frac{\partial\rho}{\partial\Theta}\frac{d\Theta}{ds} + \frac{\partial\rho}{\partial S}\frac{dS}{ds} + \frac{\partial\rho}{\partial p}\frac{dp}{ds}. \tag{19}$$

In this case, it rephrases as

$$\frac{d\rho}{dz} = \frac{\partial\rho}{\partial\Theta}\frac{d\Theta}{dz} + \frac{\partial\rho}{\partial S}\frac{dS}{dz} + \frac{\partial\rho}{\partial p}\frac{dp}{dz}. \tag{20}$$

Equations (16), (18) and (20) give the correct Brunt-Vaisalla frequency stated in eq.(11). If we choose the adiabatic path, slow enough compared to compressibility forces ($\frac{dz}{dt} << c_s$, where c_s is the medium's sound speed), such that the inner pressure from the TMV always balances the medium's pressure, then $\frac{D\tilde{p}}{Dt} = \frac{dp}{dt}$ would hold, and eq.(18) would simplify under the hydrostatic approximation to

$$\frac{d\tilde{\rho}}{dz} = \frac{\partial\tilde{\rho}}{\partial p}\frac{dp}{dz} = -\frac{\rho g}{c_s^2} \tag{21}$$

where c_s is the speed of sound in the TMV. It is, approximately, the speed of sound in the medium.
The Brunt-Vaisalla frequency then writes

$$N^2 = -\frac{g}{\rho}\frac{\partial \rho}{\partial z} + \frac{g^2}{c_s^2} \tag{22}$$

which is the same as equation (5). Q.E.D.
An alternative demonstration was already described by Mellor in the appendix of *POM user's guide* [6].

5 Discussion

The assumptions are the Archimedes principle and the adiabatic, isohaline perturbation with pressure equilibrium along its path. The hydrostatic approximation isn't required, as is for Mellor, though it's present implicitly in the density EOS. The concept is clear and simple. It clearly implies that *in-situ* temperatures need to be corrected for potential ones, because of the adiabatic lapse rate effect. Also the use of the potential density without pressure correction seems indeed a very good approximation, as was already pointed out by Mellor. Of all the codes available for the calculation of the Brunt-Vaisalla frequency (GOTM, ROMS, POM, MOM4, MOHID), the GOTM's approach seems the more direct-to-the-physics. Because it is so simple it is probably the one that introduces less round off errors. The others relate to the alternate equations reviewed in this work.

References

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